

Braids for pretzel links

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Abstract

We give a general procedure that provides, given any particular pretzel link, a braid whose closure is the pretzel link. Moreover, we manage to give a specific braid word in terms of the entries of the pretzel link.

Keywords: *Pretzel link, closed braid, Seifert circle, reducing move, complexity.*

MSC Class: *57M25.*

1 Introduction

Given integers a_1, \dots, a_n , denote by $P(a_1, \dots, a_n)$ the pretzel link diagram shown in Figure 1. Here a_i indicates $|a_i|$ crossings, with signs $a_i/|a_i|$ if $a_i \neq 0$ (we follow the notation in [4]).

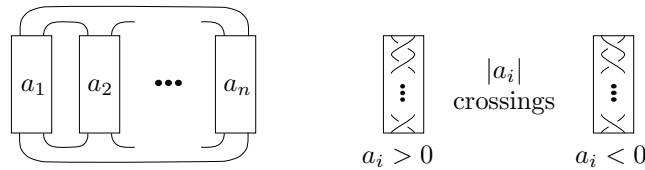


Figure 1: Pretzel link diagram $P(a_1, \dots, a_n)$.

A pretzel link is a link that has a pretzel diagram. It is not difficult to see that P represents a link with more than one component if there are two or more even entries, or if there is an even number of entries and all of them are odd (n is even and each a_i is odd).

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There are many algorithms for calculating polynomial invariants for a link, from a braid whose closure is the link (see for example the programs by Morton and Short [6]). The necessity of computing polynomial invariants for pretzel links was the original motivation for trying to find, from any pretzel link $P = P(a_1, \dots, a_n)$, a braid β whose closure is P , i.e., $P = \hat{\beta}$. Some particular cases of these braids can be found in [8] and [5]. The paper [7] addresses this issue but solves just a few examples.

Alexander [1] proved in 1923 that every link is the closure of a braid. However, his strategy is geometric and does not provide a practical algorithm for finding the braid. A more combinatorial (and recent) argument can be deduced from the work by Shuji Yamada [11], Pierre Vogel [10] and Pawel Traczyk [9]. Peter Cromwell nicely explains in his book [2] the proof given by Pawel Traczyk [9], based on the so called reducing move shown in Figure 2.

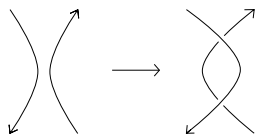


Figure 2: A reducing move. The two arcs on the left must belong to different Seifert circles.

In general, any diagram can be converted in a braid diagram performing some reducing moves. The complexity of an oriented diagram is the number of incompatible pairs of oriented Seifert circles, where two oriented disjoint circles in the sphere are said to be incompatible if both circles have the orientation inherited from an orientation in the annulus which is a cobordism of them (a less sophisticated definition is the following: two oriented Seifert circles are incompatible if, seen as gears that rotate according to their orientations, are incompatible). A diagram is a braid diagram if and only if its complexity is zero. Then one can check that each reducing move reduces the complexity of the diagram by one, and that if the complexity is greater than zero, then it is possible to apply a reducing move.

However, the direct application of this strategy over the pretzel diagrams drives us quickly to an intricate mess. This has obligated us to develop a couple of new moves on a pretzel diagram, identified in the paper as *basic move on a column* and *column shift*. These two moves, together with a multiple reducing move, will be the tools for obtaining our algorithms.

The obtained braids are far from being minimal respect to the number of crossings. However, we have checked in many cases that they are minimal respect to the number of strands. Theorem 2 provides a braid with 4 strands for the pretzel knot $P(1, 1, 1, -2)$ although its minimal braid number is 3; nevertheless one should note that this pretzel knot can be written with only three entries.

The paper is organized as follows: in Section 2 we introduce the two special types of moves, to be applied on pretzel diagrams. Section 3 contains the general strategy to go from any arbitrary pretzel diagram to the braid diagram, distinguishing whether there is an even number of entries (Theorem 2) or an odd number of entries (Theorem 3). In both cases specific braid words for each pretzel diagram are displayed; these words are in some cases a bit complicated, and needs some codification that we provide with the necessary notation. The case of three entries is shown apart for its special relevance (Theorem 1).

2 Special moves

We first introduce the special moves that will be applied on the pretzel diagram in order to obtain the wanted braid. A basic move (in a column of crossings) takes the over (under) segment of a crossing of a column and take it over (under) the rest of the column. Figure 3 shows a basic move of the third over segment. Note that a basic move increases the number of crossings by one except if it is applied on the last crossing, in which case it has no effect at all.

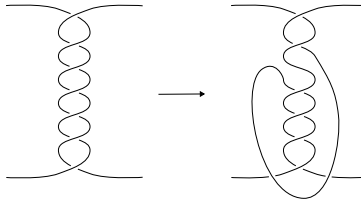


Figure 3: A basic move.

If a column has its two strands oriented in opposite directions, applying appropriate basic moves gives a nice configuration of concentric Seifert circles.

Precisely, if the number $|a|$ of crossings of the column is odd, performing $\frac{|a|-1}{2}$ appropriate basic moves we obtain exactly $\frac{|a|-1}{2}$ concentric Seifert circles. The basic moves can be performed on the crossings in odd positions (first, third and so on until the antepenultimate), or in even positions (second, fourth, etc.), as Figure 4 shows in the case $a = 7$. In a Seifert circles diagram, a colored segment (we call it a scar) recalls us where a crossing was: it is colored green if the crossing was positive, red if negative.

If $|a|$ is even, performing $\frac{|a|}{2}$ appropriate basic moves we obtain exactly $\frac{|a|}{2}$ concentric Seifert circles. Once more, the basic moves can be performed on the crossings in odd positions obtaining $\frac{|a|}{2}$ extra crossings, or in even positions obtaining $\frac{|a|}{2} - 1$ extra crossings (the last basic move has no effect at all), as Figure 5 shows in the case $a = 8$.

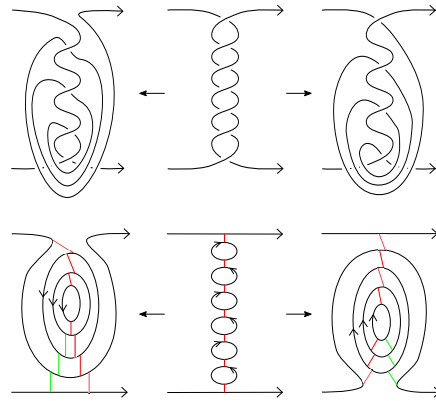


Figure 4: Basic moves on a column with an odd number $|a|$ of crossings (here $a = 7$). The top central picture is the original column of crossings. Basic moves on the odd (even) crossings produce the picture on the left (right). How it affects the Seifert circles is shown in the bottom line.

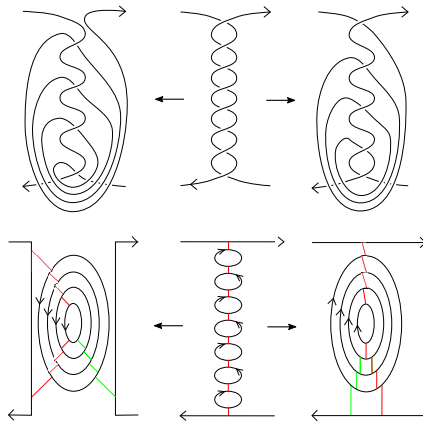


Figure 5: Basic moves on a column with an even number $|a|$ of crossings (here $a = 8$). The top central picture is the original column of crossings. Basic moves on the odd (even) crossings produce the picture on the left (right). How it affects the Seifert circles is shown in the bottom line.

The following remarks are valid whether $|a|$ is even or odd (Figures 4 and 5). First, note that if we reverse the orientation of the two strands, then the orientation of the Seifert circles must be reversed in both initial and final Seifert circle diagrams. Second, in both Figures 4 and 5 we show cases in which $a > 0$; if $a < 0$ we must then isotopy the under segments: the oriented Seifert circles remain the same, but the scars must interchange their color in both initial and final Seifert circle diagrams.

The basic move can be seen as an intra-column move. We will now describe an inter-column version of it. A *column shift* is shown in Figure 6. The chosen column is moved out to the left by a π -rotation over the paper around a vertical axis situated between the first and second columns. This creates four new crossings, two with the top horizontal line, two with the bottom horizontal line. Note that out of the pretzel diagram we obtain an exact copy of the shifted column, keeping the same *unoriented* sign of the crossings of the original column.

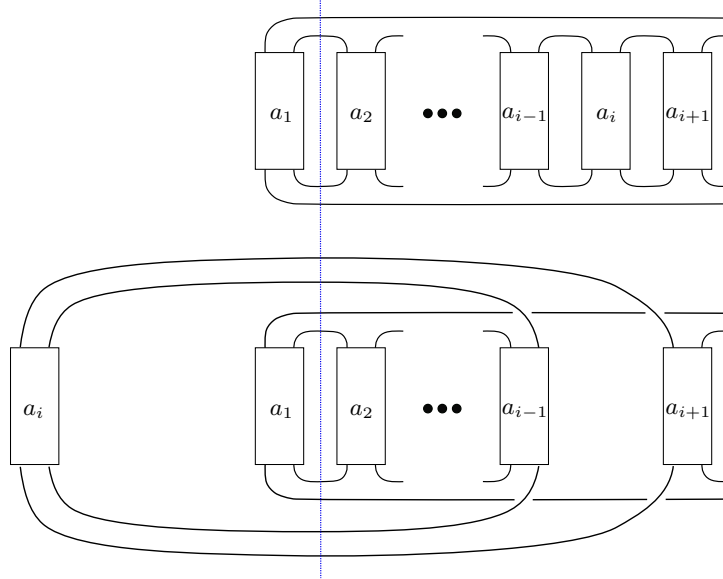


Figure 6: Shift of the i th-column.

Finally, in this Section we introduce some notation that will make easier the reading of the braid words that will appear later. As usual, the sequence of non-zero integers (m_1, \dots, m_k) codes the braid word $\sigma_{|m_1|}^{\epsilon_1} \dots \sigma_{|m_k|}^{\epsilon_k}$ where $\epsilon_i = |m_i|/m_i$ is the sign of the integer m_i . Of course, $(m_1, \dots, m_r)(m'_1, \dots, m'_s) = (m_1, \dots, m_r, m'_1, \dots, m'_s)$. It will also be convenient to have a specific notation for some types of sequences of non-zero integers. If i and j are integers, $[i \uparrow j]$ will denote the sequence of integers $(i, i+1, \dots, j-1, j)$ if $i \leq j$ and $ij > 0$; otherwise it will denote the empty braid word. For example, $[-5 \uparrow -3] = (-5, -4, -3) = \sigma_5^{-1} \sigma_4^{-1} \sigma_3^{-1}$ but $[-3 \uparrow -5] = \emptyset$ or $[-3 \uparrow 5] = \emptyset$. Analogously it is considered the notation $[i \downarrow j]$. Also, if a is a non-zero integer, we denote its sign $|a|/a$ by s_a and assume the equality $s_a(m_1, \dots, m_k) = (s_a m_1, \dots, s_a m_k)$. Then $s_a[i \uparrow j]$ is $[i \uparrow j]$ if $a > 0$ and $[-i \downarrow -j]$ if $a < 0$. Analogously, $s_a[i \downarrow j]$ is $[i \downarrow j]$ if $a > 0$ and $[-i \uparrow -j]$ if $a < 0$. Finally, $[i \uparrow j]^{\downarrow m \downarrow}$ where m is a non-negative integer means $[i \uparrow j][i-1 \uparrow j-1][i-2 \uparrow j-2] \dots$ until we complete

m groups (in particular it is the empty braid word if $m = 0$). Precisely,

$$[i \uparrow j]^{\downarrow m \downarrow} = [i \uparrow j][i - 1 \uparrow j - 1] \dots [i - (m - 1) \uparrow j - (m - 1)].$$

Also reversing arrows we have, for example, $[2 \uparrow 4]^{\uparrow 3 \uparrow} = (2, 3, 4, 3, 4, 5, 4, 5, 6)$ or $[-3 \uparrow -1]^{\uparrow 2 \uparrow} = [-3 \uparrow -1][-2 \uparrow 0] = (-3, -2, -1)$.

3 Results

The pretzel diagram $P(a)$ corresponds to the trivial knot. Also, $P(a, b) = \hat{\beta}$ where β is the braid with two strands $\beta = \sigma_1^{a+b}$. In the following theorem we will examine the case of three entries. Since we can present a pretzel diagram in a circular fashion [3], we can see its entries in a cyclic fashion. We will refer to this fact as the cyclic property of the pretzel diagrams.

Theorem 1. *Let $P = P(a, b, c)$ be a pretzel diagram with three entries. Then $P = \hat{\beta}$ where:*

1. *If there is at least one even entry, that we may assume to be the central one b by the cyclic property, then*

$$\beta = 1^c s_b[-2 \downarrow -\frac{|b|}{2} - 1] 1^a s_b \left([2 \uparrow \frac{|b|}{2}] [-\frac{|b|}{2} - 1 \uparrow -2] \right).$$

2. *If there is no even entries at all,*

$$\begin{aligned} \beta = & [b' + c' + 2 \downarrow b' + 3]^{\uparrow a' \uparrow} \\ & s_a([-b' - 2 \downarrow -b' - a' - 1][b' + 1 \uparrow b' + a'][-b' - a' - 1 \uparrow -b' - 1]) \\ & [-b' - a' - 2 \uparrow -b' - 3]^{\downarrow c' \downarrow} \\ & s_b([-b' \uparrow -1][-1 \downarrow -b' - 1][1 \uparrow b']) \\ & s_c([-b' - 2 \downarrow -b' - c' - 1][b' + 1 \uparrow b' + c'][-b' - c' - 1 \uparrow -b' - 1]) \end{aligned}$$

where

$$a' = \frac{|a| - 1}{2}, \quad b' = \frac{|b| - 1}{2} \quad \text{and} \quad c' = \frac{|c| - 1}{2}.$$

In the first case β has $|a| + |c| + \frac{3|b|}{2} - 1$ crossings and $\frac{|b|}{2} + 2$ strands. In the second case β has $|a| + |c| - 1 + \frac{3|b| + |ac|}{2}$ crossings and $\frac{|a| + |b| + |c| + 1}{2}$ strands.

Proof. Suppose that there is at least one even entry. By the cyclic property we may assume that the central entry is even, and orientation can be chosen in such a way that the strands of the central column run in opposite directions, and the top horizontal line goes to left, as shown in Figure 7.

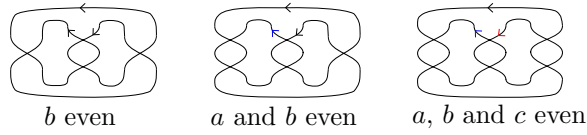


Figure 7: Orientation for the pretzel $P(a, b, c)$ when at least one entry is even.

Then we apply $\frac{|b|}{2}$ basic moves on the *even* crossings of the central column, adding $\frac{|b|}{2} - 1$ crossings (since the last basic move is redundant). The result is (apply some isotopies if you want to see all the circles concentric in the plane, not just in the sphere) a set of $\frac{|b|}{2} + 2$ Seifert circles with complexity zero (see Figure 8). It is easy to check that in all these cases the braid word is as stated above; note that the crossings in the braid corresponding to the first entry have the same sign as a , and the same happens for the crossings in the third column of the pretzel diagram (see Figure 8).

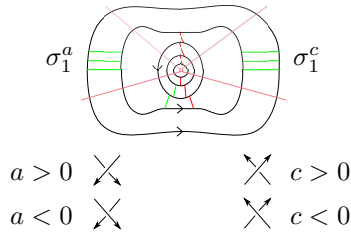


Figure 8: Braid for the pretzel $P(a, b, c)$ when at least one entry is even. A scar would have the opposite color if the corresponding entry were negative. Below: the crossings in the braid word corresponding to the extreme entries a and c have the signs of a and c respectively.

Now, we consider the case in which the three entries are odd, none of them equal to one. For convenience, orient the knot with the top segment running to the left (see Figure 9, left side). Apply now $a' = (|a| - 1)/2$ basic moves over the odd crossings of the first column, $b' = (|b| - 1)/2$ basic moves over the even crossings of the central column and again $c' = (|c| - 1)/2$ basic moves over the odd crossings of the third column. What we obtain (when seen in a sphere) are three groups of concentric Seifert circles, being those in the middle (a total of $2 + b'$) compatible with those in the extremes (a total of $a' + c'$), as shown on the right side of Figure 9.

In a second step, we use multiple reducing moves among the circles of both extremes, a' on the left, c' on the right. This produces a total of $2a'c'$ new crossings but the number of Seifert circles remains unchanged. This provides a set of $a' + b' + c' + 2$ Seifert circles with complexity zero, therefore a braid diagram with $a' + b' + c' + 2 = \frac{|a| + |b| + |c| + 1}{2}$ strands. The total number of crossings is



Figure 9: Orientation for the pretzel diagram $P(a, b, c)$ when a, b and c are odd, and the Seifert circles obtained after applying basic moves.

$|a| + a' + |b| + b' + |c| + c' + 2a'c' = |a| + |c| - 1 + \frac{3|b|+|ac|}{2}$ and the obtained braid word is as stated (details in the general case of an odd number of entries will be given in the proof of Theorem 3). If an extreme entry is equal to 1, we do not need the second step. If $b = 1$, we do not need to apply basic moves in the central column; in any case, the formula remains to be correct. ■

Theorem 2. *Let $P = P(a_1, a_2, \dots, a_n)$ be a pretzel diagram with an even number $n > 2$ of entries. Then $P = \hat{\beta}$ where*

$$\beta = \begin{bmatrix} 1 \uparrow (n-2) \\ (n-1)^{a_1} (n-3)^{a_3} \dots 1^{a_{n-1}} \\ [-(n-2) \uparrow -1] \\ (n-1)^{a_2} (n-3)^{a_4} \dots 1^{a_n} \end{bmatrix}$$

The braid β has $2n + \sum_{i=1}^n |a_i| - 4$ crossings and n strands.

Proof. The pretzel diagram defines a knot if it has exactly one even entry. Otherwise we are dealing with a link. In any case we can orient the pretzel diagram with the top segment going to left, and in each column the two strands have the same orientation, as shown on the left of Figure 10. The corresponding Seifert circles can be seen on the right of the same figure.

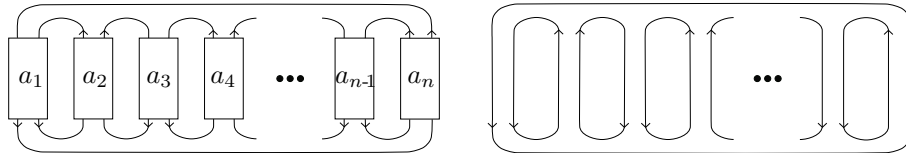


Figure 10: Chosen orientation if n is even, and the corresponding Seifert circles.

We then use $\frac{n}{2} - 1$ column shifts, applied to the columns $3, 5, \dots, n-1$. Figure 11 shows how to do it.

The result are n columns with entries, from left to right, $a_{n-1}, a_{n-3}, \dots, a_3, a_1, a_2, a_4, \dots, a_{n-2}, a_n$, and 4 extra crossings for each displaced column, two with the top horizontal segment, two with the bottom horizontal segment, therefore a

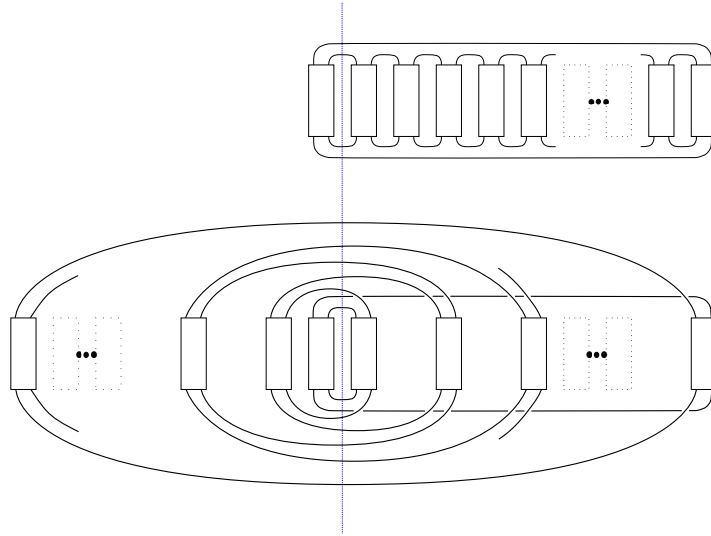


Figure 11: Column shifts of the columns $3, 5, \dots, n - 1$, where n is even.

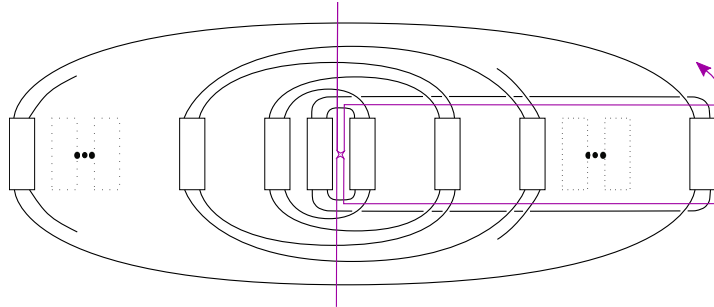


Figure 12: The braid when n is even, divided in four braid subwords.

total of $\sum_{i=1}^n |a_i| + 4(\frac{n}{2} - 1) = 2n + \sum_{i=1}^n |a_i| - 4$ crossings. The configuration of the Seifert circles has then complexity equal to zero, and in the braid word the two groups of crossings with both horizontal segments provide the braid words $[1 \uparrow (n - 2)]$ and its inverse. There are also two other groups (see Figure 12), that correspond to the second and four lines in the statement. The number of strands is n . ■

Theorem 3. Let $P = P(a_1, a_2, \dots, a_n)$ be a pretzel diagram with an odd number n of entries. Then $P = \hat{\beta}$ where

1. If there is at least one even entry, that we may assume to be a_1 by the cyclic property, then

$$\begin{aligned} \beta &= 1^{a_{n-1}} 3^{a_{n-3}} \dots (n-2)^{a_2} \\ &\quad s_{a_1} [1-n \downarrow 2-n - \frac{|a_1|}{2}] \\ &\quad [2-n \uparrow -1] \\ &\quad 1^{a_n} 3^{a_{n-2}} \dots (n-2)^{a_3} \\ &\quad [1 \uparrow n-2] \\ &\quad s_{a_1} ([n-1 \uparrow n-3 + \frac{|a_1|}{2}] [2-n - \frac{|a_1|}{2} \uparrow 1-n]). \end{aligned}$$

2. If there is no even entries at all,

$$\beta = G(n) \dots G(3)G(2)T(1)G(2)^{-1}T(2)G(3)^{-1} \dots T(n-1)G(n)^{-1}T(n)$$

where, for $j = 1, \dots, n$

$$\begin{aligned} T(j) &= \epsilon_j ([-2 \downarrow -b_j - 1][1 \uparrow b_j][-b_j - 1 \uparrow -1]) \\ &= \sigma_2^{-\epsilon_j} \sigma_3^{-\epsilon_j} \dots \sigma_{b_j+1}^{-\epsilon_j} (\sigma_1^{\epsilon_j} \sigma_2^{\epsilon_j} \dots \sigma_{b_j}^{\epsilon_j}) \sigma_{b_j+1}^{-\epsilon_j} \dots \sigma_2^{-\epsilon_j} \sigma_1^{-\epsilon_j} \end{aligned}$$

being $b_j = \frac{|a_j|-1}{2}$ and $\epsilon_j = \frac{a_j}{|a_j|}$ the sign of the entry a_j , and for $k = 2, \dots, n$

$$G(k) = G(k-1, k) \dots G(2, k)G(1, k),$$

where for $1 \leq i < j \leq n$

$$G(i, j) = [-B(i, j) - b_j \downarrow -B(i, j) - b_j - b_i + 1]^{\uparrow b_j \uparrow}$$

being $B(i, j) = 2 + b_{i+1} + \dots + b_{j-1}$, $1 \leq i < j \leq n$.

In the first case β has $\frac{|a_1|}{2} + \sum_{j=1}^n |a_j| + 2n - 5$ crossings and $\frac{|a_1|}{2} + n - 1$ strands. In the second case β has $\frac{1}{2} \left(3 \sum_{j=1}^n |a_j| + \sum_{1 \leq i < j \leq n} (|a_i| - 1)(|a_j| - 1) - n \right)$ crossings and $\frac{1}{2} \left(4 - n + \sum_{j=1}^n |a_j| \right)$ strands.

Proof. Assume first that there is at least one even entry. By the cyclic property we can assume that a_1 is even. We say that a column is N-oriented (S-oriented) if its two strands are oriented from bottom to top (from top to bottom); otherwise we say that the column is not well oriented. We claim that we can orient the pretzel diagram in such a way that the columns in even position are N-oriented and those in odd positions are S-oriented, except the first one, that is not well oriented (see Figure 13).

To prove that such an orientation is possible we proceed by induction on the number k of even entries. The case $k = 1$ is left to the reader as an exercise.

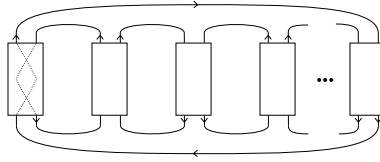


Figure 13: How to orient the pretzel diagram when n is odd and a_1 is even.

Assume now that there are other even entries, and let j be the first even entry after the first one. Compare now the pretzel diagrams $P = P(a_1, \dots, a_n)$ and $P' = P(a_1, \dots, a_{j-1}, 1 + a_j, a_{j+1}, \dots, a_n)$ where P' has an extra crossing in the column j . By induction we may assume the wished orientation in P' . Now this orientation can be carried to P by choosing the appropriate orientation of the extra component of P situated between the columns 1 and j (see Figure 14).

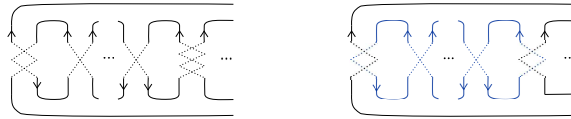


Figure 14: Carrying the orientation from P' to P .

We now apply $\frac{|a_1|}{2}$ basic moves on the even crossings of the first column, obtaining $\frac{|a_1|}{2}$ concentric Seifert circles and $\frac{|a_1|}{2} - 1$ extra crossings, as shown in Figure 15. The color of the scars is the opposite (red are green and vice versa) if $a_1 < 0$.

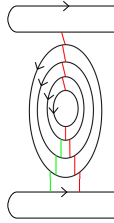


Figure 15: Basic moves in the first column of $P(a_1, \dots, a_n)$, n odd and a_1 even. Color of each scar is the opposite if $a_1 < 0$. Here $a_1 = 8$.

We now apply column shifts to the columns $3, 5, \dots, n$, as shown in Figure 16.

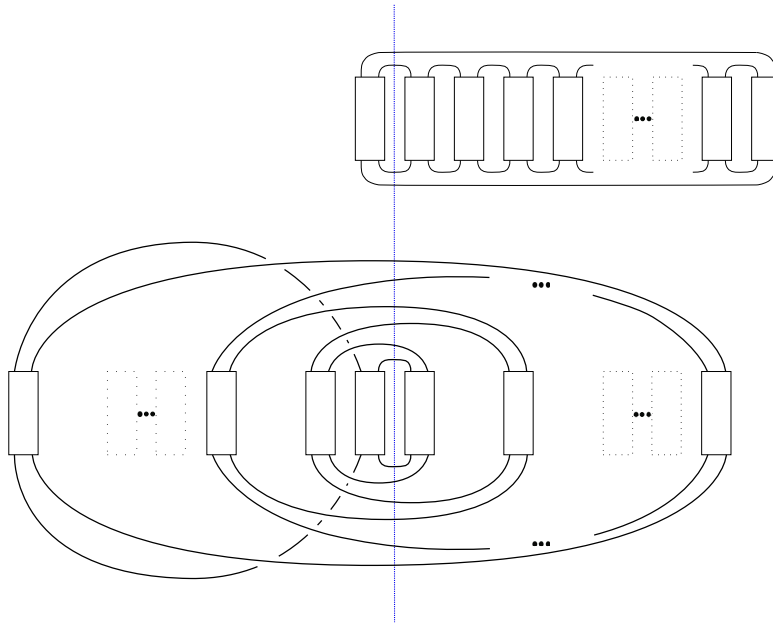


Figure 16: Column shifts of the columns $3, 5, \dots, n$, where n is odd and a_1 even.

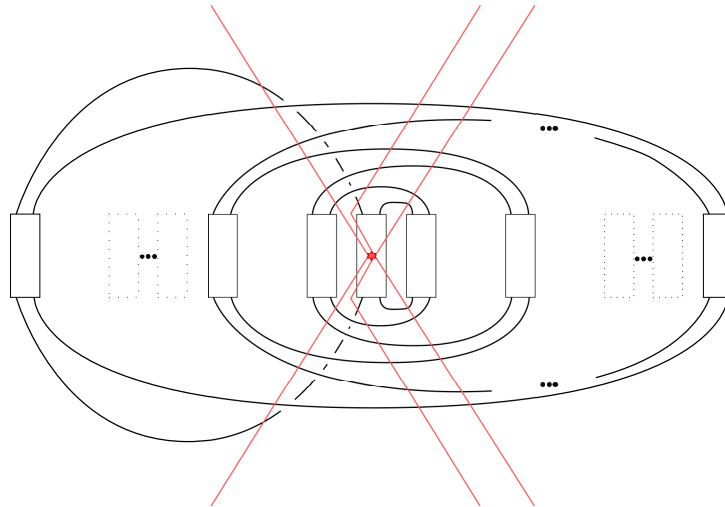


Figure 17: The braid when n is odd and a_1 is even, divided in six braid words.

We obtain a total of $N = \frac{|a_1|}{2} + 2\frac{n-1}{2} = \frac{|a_1|}{2} + n - 1$ strands for the final braid, divided in six braid subwords as shown in Figure 17. These six subwords are

the following:

$$\begin{aligned}
\beta &= 1^{a_{n-1}} 3^{a_{n-3}} \dots (n-2)^{a_2} \\
&\quad s_{a_1} \left[\frac{|a_1|}{2} - N \uparrow -(N-1) \right] \\
&\quad \left[- \left(2^{\frac{n-1}{2}} - 1 \right) \uparrow -1 \right] \\
&\quad 1^{a_n} 3^{a_{n-2}} \dots (n-2)^{a_3} \\
&\quad \left[1 \uparrow 2^{\frac{n-1}{2}} - 1 \right] \\
&\quad s_{a_1} \left(\left[(N-2) - \left(\frac{|a_1|}{2} - 2 \right) \uparrow N-2 \right] \left[-(N-1) \uparrow -(N-1) + \left(\frac{|a_1|}{2} - 1 \right) \right] \right) \\
&= 1^{a_{n-1}} 3^{a_{n-3}} \dots (n-2)^{a_2} \\
&\quad s_{a_1} \left[1 - n \uparrow 2 - n - \frac{|a_1|}{2} \right] \\
&\quad \left[2 - n \uparrow -1 \right] \\
&\quad 1^{a_n} 3^{a_{n-2}} \dots (n-2)^{a_3} \\
&\quad \left[1 \uparrow n - 2 \right] \\
&\quad s_{a_1} \left(\left[n - 1 \uparrow n - 3 + \frac{|a_1|}{2} \right] \left[2 - n - \frac{|a_1|}{2} \uparrow 1 - n \right] \right)
\end{aligned}$$

where, for the third line, it must be noted that the shift of the last column only gives one crossing on the top. This ends the proof when there is at least one even entry.

We concentrate now on the second case, when n and any entry a_i is odd. We have then a knot which can be oriented as shown in Figure 18.

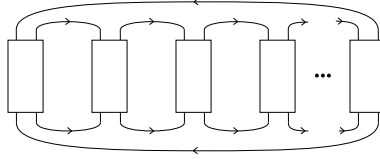


Figure 18: Orientation when n and any entry is odd.

We start by applying $b_j = \frac{|a_j|-1}{2}$ basic moves on crossings in odd position (first, third and so on until the antepenultimate) of the column j , for $j = 1, \dots, n$. In the column j we obtain b_j concentric counterclock oriented Seifert circles and b_j extra crossings. These circles and the corresponding scars (of the crossings) are shown in Figure 19 for the pretzel diagram $P(1, 3, 5, 7, 9)$. Note that the colour of the scars in the column j should be interchanged if it were $a_j < 0$.

Circles in different columns are not compatible, so we apply multiple reducing moves among all the circles in different columns. The blue box in Figure 20 bounds the space where these moves are performed. What we obtain is shown in Figure 21, where a strand labeled with a non-negative integer k means a collection of k strands that run parallel to the drawn strand, hence $\overline{\quad}^k$ means $\overline{\dots}$ (k strands). Note also that in Figure 21 a strand coming from the column j goes over the strands coming from the column i if $i < j$, hence the crossings of the top half space are negative, those of the lower half space are positive.

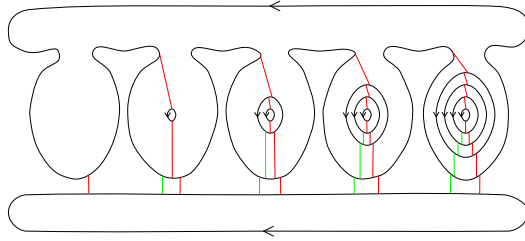


Figure 19: Result of applying $b_j = \frac{|a_j|-1}{2}$ basic moves to the odd crossings in each j th-column, when n and any a_j is odd.

In the space for multiple reducing moves we find a total of $2 \sum_{1 \leq i < j \leq n} b_i b_j = \frac{1}{2} \sum_{1 \leq i < j \leq n} (|a_i| - 1)(|a_j| - 1)$ crossings.

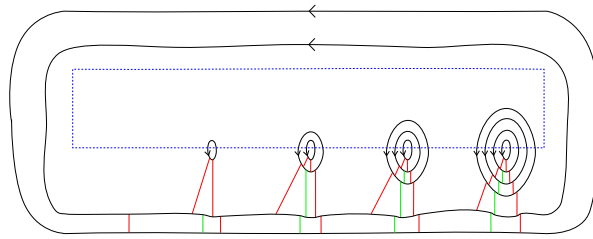


Figure 20: Space for multiple reducing moves.

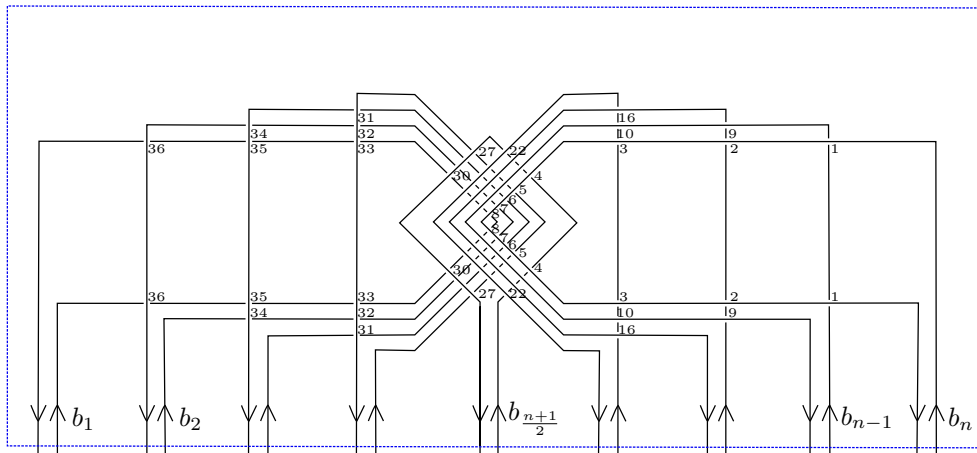


Figure 21: Multiple reducing moves.

After performing basic moves in each column and then multiple reducing moves

among arcs of different columns, we have obtained a presentation of the pretzel knot as a closed braid. The set of crossings in the braid originated by the crossing of the (b_i parallel strands represented by the) strand i (that coming from the column i) with the strand j , for $i < j$, can be ordered as a subword of the braid word, and it will be called $G(i, j)$. In particular, $G(i, j)$ is the empty braid word if $b_i = 0$ or $b_j = 0$. Moreover, the braid word can be divided globally into $3n - 2$ subwords as we show in Figure 22. The wanted braid β can be then written as

$$\beta = G(n) \dots G(3)G(2) T(1)G(2)^{-1}T(2)G(3)^{-1} \dots T(n-1)G(n)^{-1}T(n)$$

where $T(j)$ takes into account the crossings derived from the basic moves in the column j , and, following the numeration in Figure 21,

$$G(k) = G(k-1, k) \dots G(2, k)G(1, k).$$

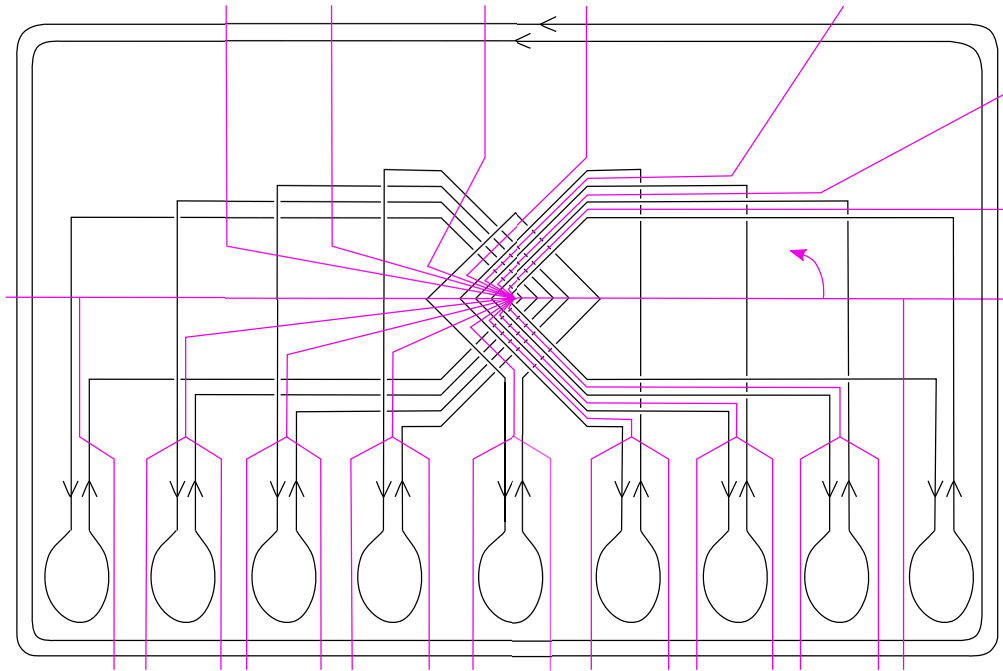


Figure 22: Mixed diagram representing the final braid: the top part is made of *non-amplified* crossings; the lower part shows partial Seifert circles (here $n = 9$). This picture allows to distinguish the different subwords in the final braid word. The oriented arc indicates where we start to write the braid word.

The braid word $G(i, j)$ is (see Figure 23)

$$\begin{aligned}
G(i, j) &= (\sigma_{B(i,j)+b_j}^{-1} \sigma_{B(i,j)+b_j+1}^{-1} \cdots \sigma_{B(i,j)+b_j+b_i-1}^{-1}) \\
&\quad (\sigma_{B(i,j)+b_j-1}^{-1} \sigma_{B(i,j)+b_j}^{-1} \cdots \sigma_{B(i,j)+b_j-1+b_i-1}^{-1}) \\
&\quad \cdots (\sigma_{B(i,j)+1}^{-1} \sigma_{B(i,j)+2}^{-1} \cdots \sigma_{B(i,j)+b_i}^{-1}) \\
&= (-B(i, j) - b_j \downarrow -B(i, j) - b_j - b_i + 1) \\
&\quad (-B(i, j) - b_j + 1 \downarrow -B(i, j) - b_j + 1 - b_i + 1) \\
&\quad \cdots (-B(i, j) - 1 \downarrow -B(i, j) - b_i) \\
&= [-B(i, j) - b_j \downarrow -B(i, j) - b_j - b_i + 1]^{\uparrow b_j \uparrow}
\end{aligned}$$

where $B(i, j) = 2 + b_{i+1} + \cdots + b_{j-1}$ is the number of strands that surround the outside of the crossing. Here it must be understood from the notation that $G(i, j)$ is the empty word if $b_i = 0$, that is, if $a_i = \pm 1$. But by definition $G(i, j)$ is also the empty word if $b_j = 0$. Note also that one could read $G(i, j)$ following the order of the crossings 1, 4, 7, 10; 2, 5, 8, 11; 3, 6, 9, 12 in Figure 23.

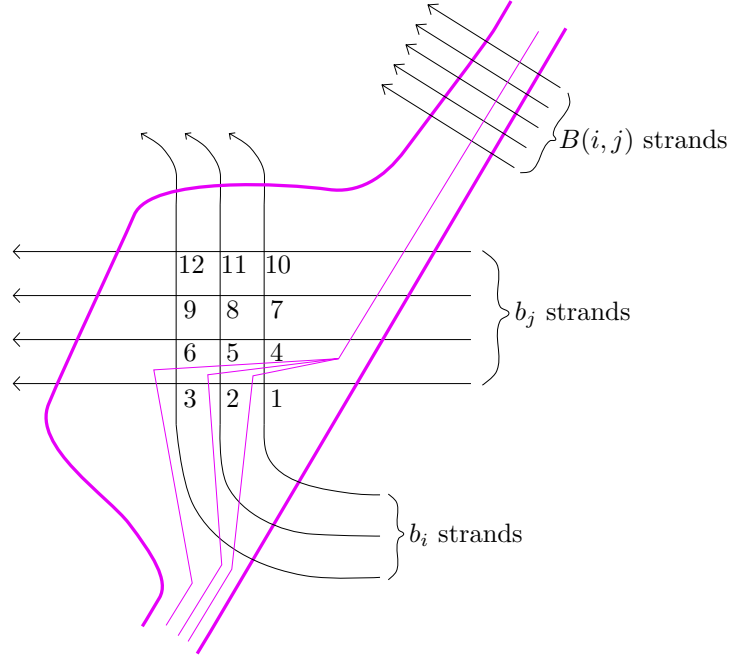


Figure 23: Detail of the multiple reducing move: amplified crossing of Figure 21.

The subword $T(j)$ is

$$\begin{aligned}
T(j) &= (-2 \downarrow -b_j - 1)(1 \uparrow b_j)(-b_j - 1 \uparrow -1) \\
&= \sigma_2^{-1} \sigma_3^{-1} \cdots \sigma_{b_j+1}^{-1} (\sigma_1 \sigma_2 \cdots \sigma_{b_j}) \sigma_{b_j+1}^{-1} \cdots \sigma_2^{-1} \sigma_1^{-1}
\end{aligned}$$

where the central parenthesis corresponds to the green crossings as can be seen in any column in Figure 20 (colors must be interchanged if the corresponding

$a_j < 0$). Note that $T(j) = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ if $b_j = 1$, that is, $a_j = \pm 3$, and $T(j) = \sigma_1^{-1}$ if $b_j = 0$, that is, $a_j = \pm 1$. In general, in $T(j)$ there are $3b_j + 1$ crossings.

We finally count the number of crossings and strands in the obtained braid. The number c of crossings is

$$\begin{aligned}
c &= \sum_{j=1}^n (3b_j + 1) + 2 \sum_{1 \leq i < j \leq n} b_i b_j \\
&= n + 3 \sum_{j=1}^n b_j + 2 \sum_{1 \leq i < j \leq n} b_i b_j \\
&= n + 3 \sum_{j=1}^n \frac{|a_j| - 1}{2} + 2 \sum_{1 \leq i < j \leq n} \frac{|a_i| - 1}{2} \frac{|a_j| - 1}{2} \\
&= -\frac{n}{2} + \frac{3}{2} \sum_{j=1}^n |a_j| + \frac{1}{2} \sum_{1 \leq i < j \leq n} (|a_i| - 1)(|a_j| - 1)
\end{aligned}$$

and the total number b of strands is

$$b = 2 + \sum_{j=1}^n b_j = 2 + \sum_{j=1}^n \frac{|a_j| - 1}{2} = 2 + \frac{1}{2} \left(\sum_{j=1}^n |a_j| - n \right) = \frac{1}{2} \left(4 - n + \sum_{j=1}^n |a_j| \right).$$

■

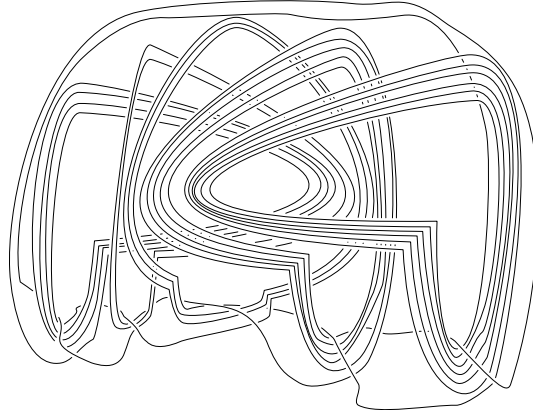


Figure 24: The pretzel knot $P(9, 5, 7, 11, 13)$ as closed braid.

As an example, we show in Figure 24 the braid (with 22 strands and 375 crossings) for the pretzel knot $P(9, 5, 7, 11, 13)$, according to the second point of Theorem 3. You can quickly find a braid whose closure is your favourite pretzel link in the following web site, in which the first author has implemented the algorithms described in this paper: <https://adelpozoman.es/braid/pretzel/>

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